Lecture 3 Utility Maximizations

ECON201D

Luke Zhao

Based on: *Microeconomics: An Intuitive Approach with Calculus, 2nd Edition,*
by Thomas Nechyba

The Consumer's Problem

The consumer's problem is modelled as a constrained optimization problem:

$$\begin{array}{rlcl} \max & \text{Happiness through purchasing} & \Rightarrow & \text{The utility function} \\ \text{subject to} & \text{Exogenous economic circumstances} & \Rightarrow & \text{The budget set.} \end{array}$$

Formally, let p_i be the prices, I be the income, the consumer solves

$$\begin{array}{rl} \max & u(x_1, x_2, \cdots, x_n), \\ \text{subject to} & p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = I. \end{array}$$

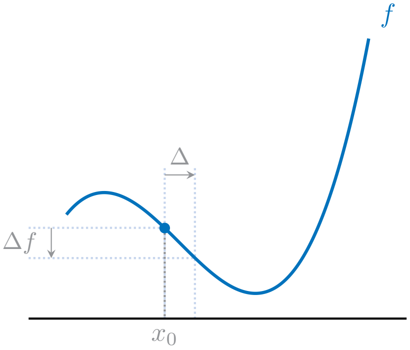
Derivatives

The *derivative* of the function $f : \mathbb{R} \to \mathbb{R}$ at $x = x_0$, denoted by $f'(x_0)$, is

$$f'(x_0) = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0} = \lim_{\Delta \to 0} \frac{f(x_0 + \Delta) - f(x_0)}{\Delta}.$$

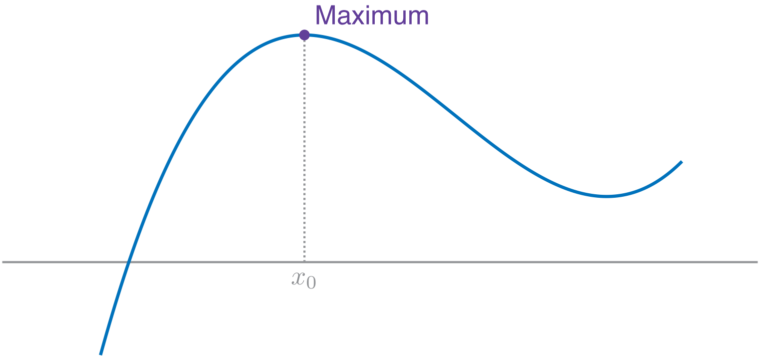
In words, this derivative measures how much the value $f(x)$ of the function changes as the variable x changes around x_0.

- $f'(x_0) > 0$: f increases at x_0.
- $f'(x_0) < 0$: f decreases at x_0.



Optimization with Derivatives

The derivative should be 0 at maximum/minimum points: $f'(x^{\star}) = 0$. This is called the *first order condition* (FOC) since it uses the first order derivatives.



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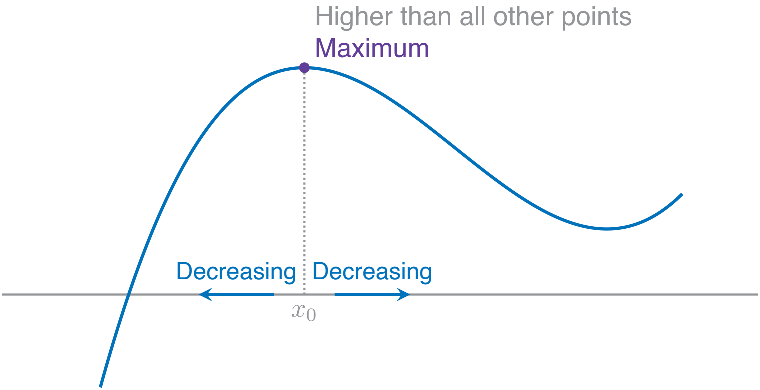

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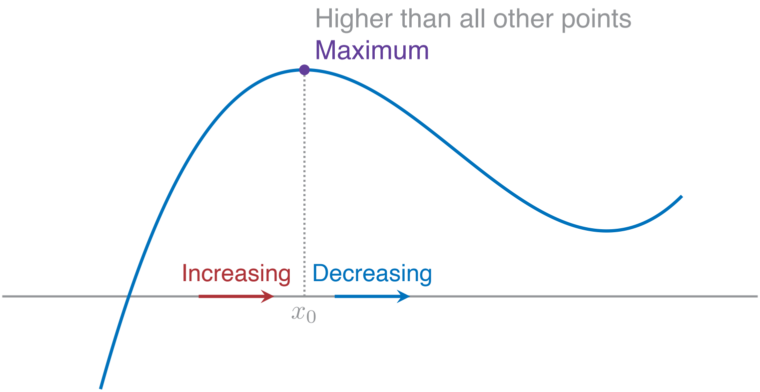

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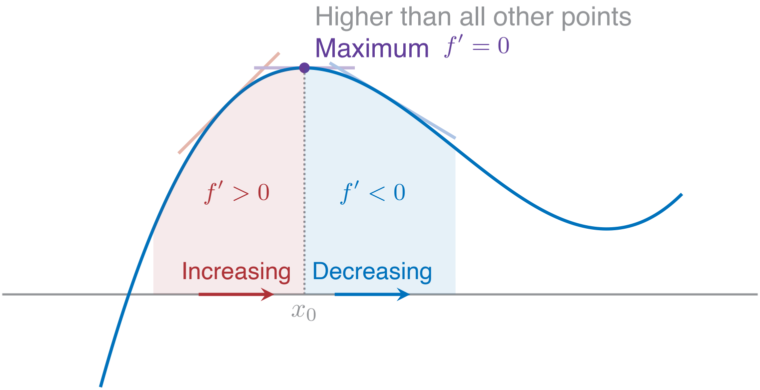

First Order Conditions

The first order conditions still hold when we have more variables. If $\mathbf{x}^\star = (x_1^\star, \cdots, x_n^\star)$ is a maximum or minimum point, then

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To show this, let's consider the maximum point and suppose the conclusion is not true. That is, suppose for some x_i, $\partial f / \partial x_i|_{\mathbf{x}=(x_1^\star, \cdots, x_n^\star)} \neq 0$.

- If $\partial f / \partial x_i|_{\mathbf{x}=(x_1^\star, \cdots, x_n^\star)} > 0$, one can keep all other $x_j = x_j^\star$ and slightly increase x_i to increase f.
- If $\partial f / \partial x_i|_{\mathbf{x}=(x_1^\star, \cdots, x_n^\star)} < 0$, one can keep all other $x_j = x_j^\star$ and slightly decrease x_i to increase f.

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First order conditions may not hold if:

- The domain of the function is restricted. For example, in terms of utility functions, x_1 and x_2 cannot be negative.
- The function (overall or at some points) is not differentiable.

First order conditions are not *sufficient* conditions unless the function has some nice properties.

Differentiable Utilities

The Lagrange Method

- For now, let's suppose that the utility functions are differentiable.

$$\begin{aligned} \max \quad & u(x_1, \cdots, x_n), \\ \text{subject to} \quad & p_1 x_1 + \cdots + p_n x_n = I. \end{aligned}$$

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The *Lagrangian* is

$$\mathcal{L}(x_1, \cdots, x_n, \lambda) = u(x_1, \cdots, x_n) + \lambda(I - p_1 x_1 - \cdots - p_n x_n).$$

λ is called a *Lagrange multiplier*.

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- FOCs are a set of $(n+1)$ equations, with $(n+1)$ unknowns.
- Take ratios of FOCs with respect to x_i and x_j:

$$|MRS(x_i, x_j)| = \frac{\partial u / \partial x_i}{\partial u / \partial x_j} = \frac{p_i}{p_j}$$

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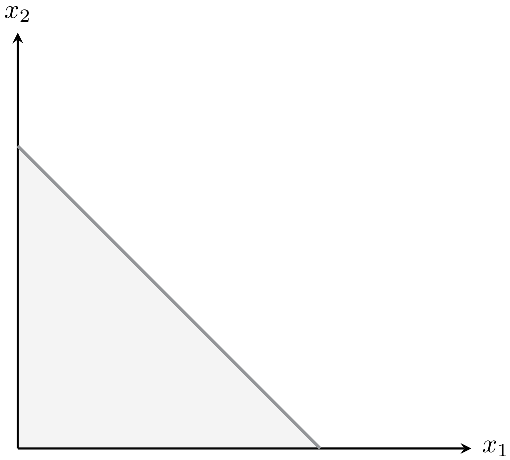
Consumer's substitution relation coincide with the market substitution relation.

Graphical Methods

Facing a given budget set…



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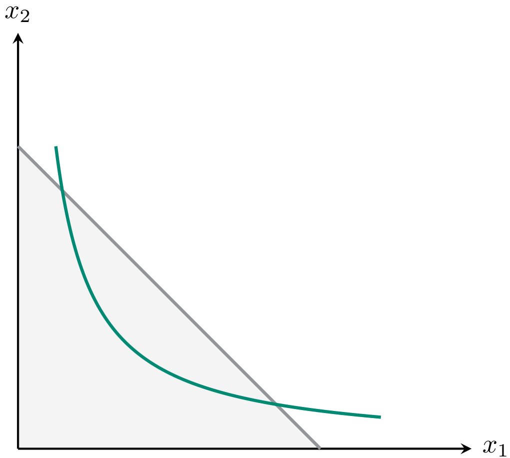
- The consumer wants to be on a higher indifference curve.



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- The consumer wants to be on a higher indifference curve.
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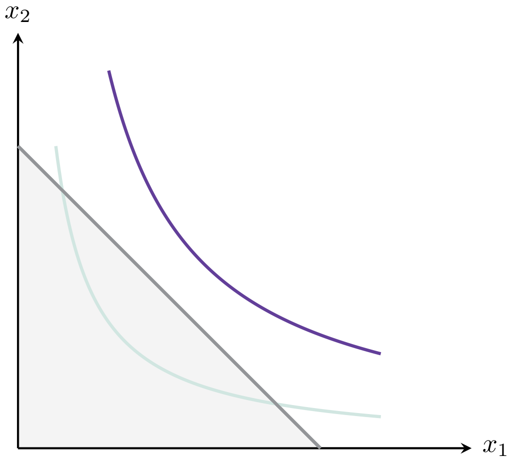

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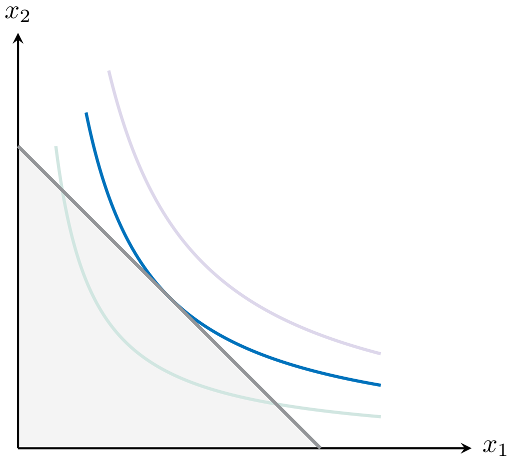
The consumer should choose the indifference curve that “just touch” the budget line – tangent if possble.



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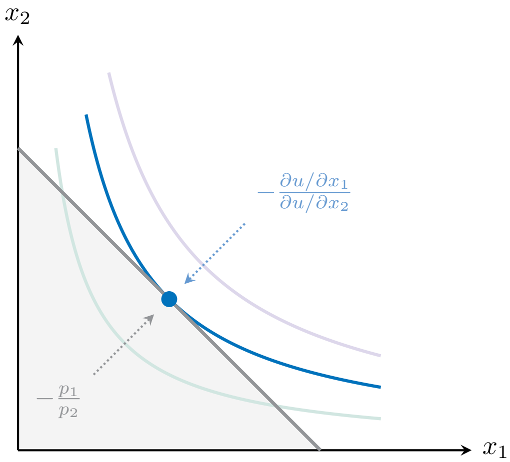
The consumer should choose the indifference curve that "just touch" the budget line – tangent if possble.

- Slope of the budget line $=$ Slope of the indifference curve:

$$|MRS(x_1, x_2)| = \frac{p_1}{p_2}.$$



Cobb-Douglas Utility Functions

A utility function (with two goods) is called a *Cobb-Douglas utility function* if

$$u(x_1, x_2) = x_1^{\alpha} x_2^{\beta},$$

where $\alpha, \beta > 0$. Use the Lagrange method to solve for the consumer's optimal bundle.

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Then get the FOCs

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \alpha x_1^{\alpha-1} x_2^\beta - \lambda p_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= \beta x_1^\alpha x_2^{\beta-1} - \lambda p_2 = 0 \\
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From the FOCs,

$$\begin{aligned} \alpha x_1^{\alpha-1} x_2^{\beta} - \lambda p_1 = 0 \quad &\Rightarrow \quad \alpha x_1^{\alpha-1} x_2^{\beta} = \lambda p_1, \\ \beta x_1^{\alpha} x_2^{\beta-1} - \lambda p_2 = 0 \quad &\Rightarrow \quad \beta x_1^{\alpha} x_2^{\beta-1} = \lambda p_2. \end{aligned}$$

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Take the ratio of the two on both sides to get a x_1–x_2 relationship,

$$\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{\beta x_1^{\alpha} x_2^{\beta-1}} = \frac{\lambda p_1}{\lambda p_2} \quad \Rightarrow \quad \frac{\alpha x_2}{\beta x_1} = \frac{p_1}{p_2} \quad \Rightarrow \quad x_2 = \frac{\beta p_1}{\alpha p_2} x_1.$$

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Substitute this relationship back to the budget constraint

$$p_1 x_1 + p_2 \frac{\beta p_1}{\alpha p_2} x_1 = I \quad \Rightarrow \quad x_1 = \frac{\alpha I}{(\alpha+\beta) p_1} \quad \Rightarrow \quad x_2 = \frac{\beta I}{(\alpha+\beta) p_2}.$$

Demand Functions

A *demand function* $x_i^\star(p_1, \cdots, p_i, \cdots, p_n, I)$ of good x_i is the consumer's choice of x_i in the optimized bundle, given the prices of all goods and the income.

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For example, for Cobb-Douglas preferences,

$$x_1^\star(p_1, p_2, I) = \frac{\alpha I}{(\alpha + \beta)p_1} \quad \text{and} \quad x_2^\star(p_1, p_2, I) = \frac{\beta I}{(\alpha + \beta)p_2}.$$

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Properties of Cobb-Douglas Demand Functions. Observe that

- The demand of $x_i^\star$ does not depend on the prices of other goods.
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Of course, these are not always the cases for other preferences / utility functions.

CES Utility Functions

The following utility function is called a *CES utility function* for $\rho < 1$. Solve the consumer's utility maximization problem with respect to this utility function.

$$u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}.$$

Hint: use a proper positive monotone transformation first.

Homothetic Preferences

Notice that

$$MRS^{\text{CD}}(x_1, x_2) = -\frac{\alpha x_2}{\beta x_1} \quad \text{and} \quad MRS^{\text{CES}}(x_1, x_2) = -\left(\frac{x_1}{x_2}\right)^{\rho-1},$$

i.e., the MRS in both case only depends on the *ratio* x_1/x_2, not the values of x_1 or x_2. The preferences can be represented by utility functions with such properties are called *homothetic preferences*.

How about the indifference curves?

Homothetic Preferences

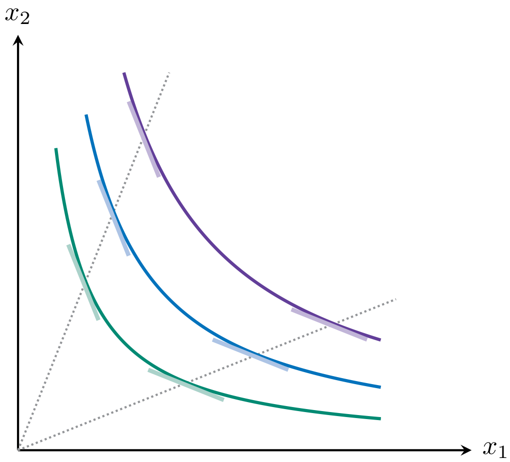
$$x_2/x_1 = k \quad \Rightarrow \quad x_2 = kx_1.$$

- x_2/x_1 is fixed on a ray passing through the origin.

Since:

- MRS is the slope of the indifference curves;
- MRS only depends on the ratio x_1/x_2;

The slopes of the indifference curves are the same along a ray from the origin.



Corner Solutions

Quasilinear Preferences

A preference relation between x_1 and x_2 is called *quasilinear in x_1* if it can be represented by a utility function

$$u(x_1, x_2) = v(x_1) + x_2,$$

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MRS of a Quasilinear Preferences. Notice that

$$MRS(x_1, x_2) = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{\partial u/\partial x_1}{1} = -\frac{\mathrm{d}v}{\mathrm{d}x_1}.$$

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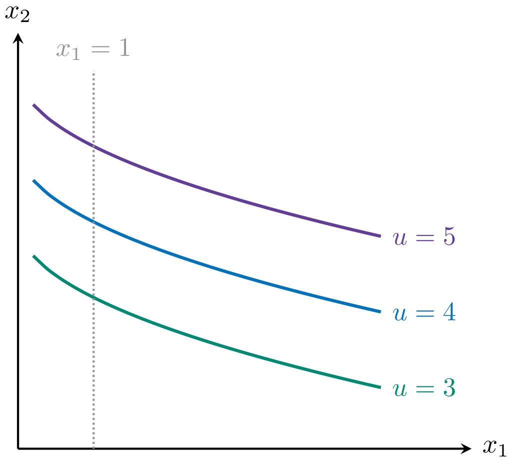
The MRS does not depend on x_2 if the preference is quasilinear in x_1.

Quasilinear Preferences

What does this mean for the shape of indifference curves? Consider the example:

$$u = \sqrt{x_1} + x_2.$$

What are the MRS at $x_1 = 1$ for different u's?



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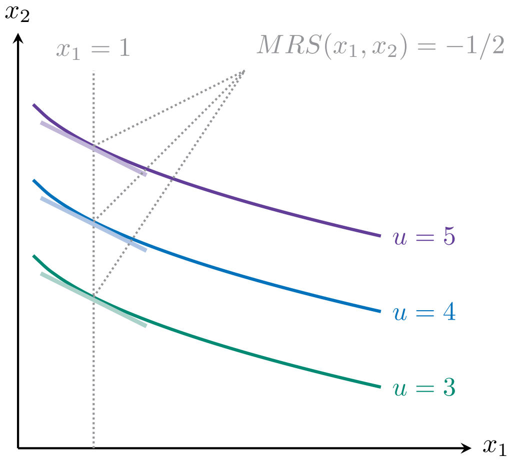

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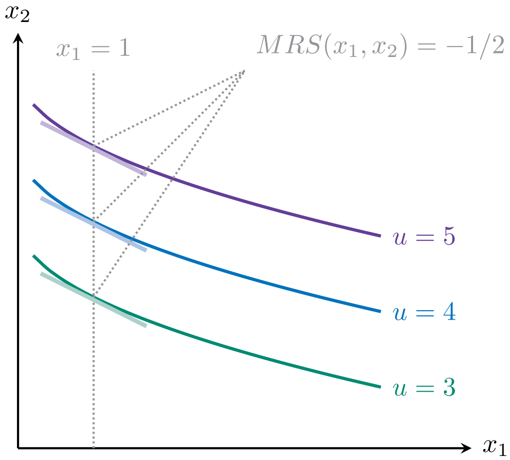

Along the vertical ray (in which x_1 in unchanged), the MRS is the same across all indifference curves.

Utility Maximizations under Quasilinear Preferences

Solve the utility maximization problem under the following utility function:

$$u(x_1, x_2) = \ln x_1 + x_2.$$

Hint: $\mathrm{d}\ln(x)/\,\mathrm{d}\,x = 1/x$.

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$$\mathcal{L}(x_1, x_2, \lambda) = \ln x_1 + x_2 + \lambda(I - p_1 x_1 - p_2 x_2).$$

And the first order conditions are

$$\begin{aligned} \frac{1}{x_1} - \lambda p_1 &= 0, \\ 1 - \lambda p_2 &= 0. \end{aligned} \quad \Rightarrow \quad \frac{1}{x_1} = \frac{p_1}{p_2} \quad \Rightarrow \quad x_1^\star = \frac{p_2}{p_1} \quad \Rightarrow \quad x_2^\star = \frac{I - p_2}{p_2}.$$

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Do you spot a problem?

Utility Maximizations under Quasilinear Preferences

Since $x_1, x_2 \geqslant 0$, the result

$$x_1^\star = \frac{p_2}{p_1} \quad \text{and} \quad x_2^\star = \frac{I - p_2}{p_2}.$$

only works if $p_2 \leqslant I$.

What happens when $p_2 > I$?

Utility Maximizations under Quasilinear Preferences

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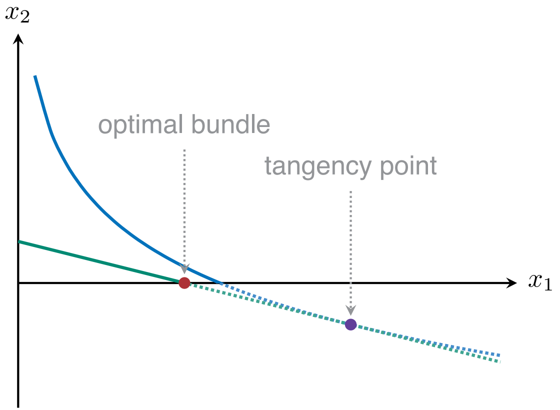
only works if $p_2 \leqslant I$.

What happens when $p_2 > I$?

- At the tangency point, $x_2 < 0$.
- The consumer would like to "purchase" $x_2 < 0$.
- But that is impossible – choose $x_2 = 0$ instead.



Utility Maximizations under Quasilinear Preferences

Why $x_2 < 0$ at the tangency point? For the utility function $u(x_1, x_2) = \ln x_1 + x_2$:

- Suppose the consumer is currently at some $(\hat{x}_1, \hat{x}_2)$ and wants to decide what to do if she has one additional dollar.
- Spending one additional dollar on x_1 or x_2 gains utility approximately

$$\frac{1}{p_1}\frac{\partial u}{\partial x_1} = \frac{1}{p_1}\frac{1}{\hat{x}_1} \quad \text{or} \quad \frac{1}{p_2}\frac{\partial u}{\partial x_2} = \frac{1}{p_2},$$

  respectively.
- When $\hat{x}_1$ is really small, $1/\hat{x}_1$ is really large, so that the gain from purchasing x_1 is much larger than the gain from purchasing x_2. The consumer gains utility if she repurposes a dollar used on x_2 to purchase x_1.
- $x_2 < 0$ at the tangency point implies the consumer wants to continue the repurposing operation even when all income is used on x_1.

Perfect Substitutions

Consider the following utility function:

$$u(x_1, x_2) = x_1 + x_2.$$

That is, increasing x_1 or x_2 increases the utilities in the same way. We call x_1 and x_2 are *perfect substitutes*.

Can you draw the indifference curves?

Perfect Substitutions

Consider the following utility function:

$$u(x_1, x_2) = x_1 + x_2.$$

That is, increasing x_1 or x_2 increases the utilities in the same way. We call x_1 and x_2 are *perfect substitutes*.

Can you draw the indifference curves?

Perfect Substitutions

Consider the following utility function:

$$u(x_1, x_2) = x_1 + x_2.$$

- Is this preference continuous?
- Is this preference monotone? Strictly monotone?
- Is this preference convex? Strictly convex?

The MRS under Perfect Substitutions

Given $u(x_1, x_2) = x_1 + x_2$, what is the MRS?

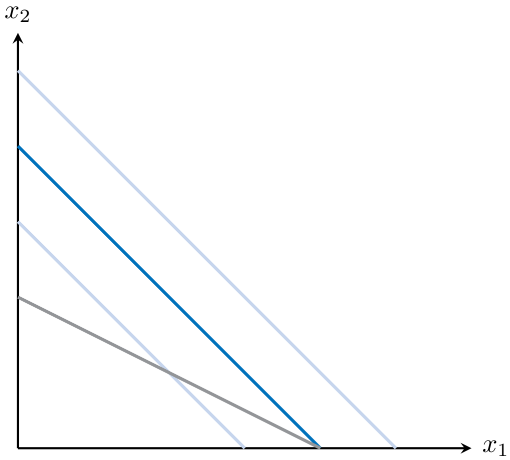
$$|MRS(x_1, x_2)| = \frac{\partial u / \partial x_1}{\partial u / \partial x_2} = \frac{1}{1} = 1.$$

- The MRS does not depend on x_1 or x_2. (Very natural, given the indifference curve is a straight line.)
- $|MRS(x_1, x_2)| = 1$: one unit of x_1 can be substituted exactly by one unit of x_2. Hence the name, perfect substitutions.
- A special case: the perfect substitution preference is both *homothetic* and *quasilinear*.

Utility Maximizations under Perfect Substitutions

Given $|MRS(x_1, x_2) = 1|$, and abs. of the slope of the budget line is p_1/p_2:

- If $p_1 < p_2$, the budget line is flatter than the indifference curves.



Utility Maximizations under Perfect Substitutions

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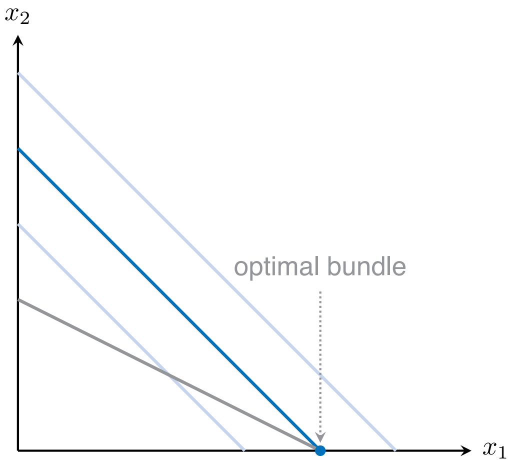
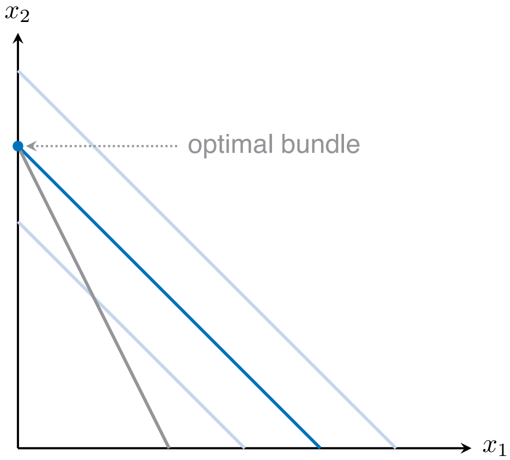
  Spend all income on x_1 only.



Utility Maximizations under Perfect Substitutions

Given $|MRS(x_1, x_2) = 1|$, and abs. of the slope of the budget line is p_1/p_2:

- If $p_1 < p_2$, the budget line is flatter than the indifference curves.
  Spend all income on x_1 only.
- If $p_1 > p_2$, the budget line is steeper than the indifference curves.
  Spend all income on x_2 only.



Utility Maximizations under Perfect Substitutions

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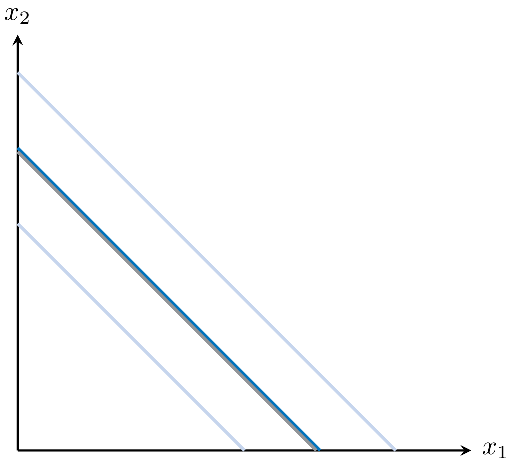
- If $p_1 < p_2$, the budget line is flatter than the indifference curves.
  Spend all income on x_1 only.
- If $p_1 > p_2$, the budget line is steeper than the indifference curves.
  Spend all income on x_2 only.
- If $p_1 = p_2$, the budget line is parallel to the indifference curves and coincides with exactly one piece.
  Optimal everywhere on the budget line.



Utility Maximizations under Perfect Substitutions

The demand “function” of x_1:

$$x_1^\star(p_1, p_2, I) = \begin{cases} I/p_1, & \text{if } p_1 < p_2, \\ [0, I/p_1], & \text{if } p_1 = p_2, \\ 0 & \text{if } p_1 > p_2. \end{cases}$$

- The consumer views x_1 and x_2 equivalently and purchases the cheaper one.
- $|MRS(x_1, x_2)| = p_1/p_2$ when $p_1 = p_2$. Otherwise, the consumer’s optimal bundle is a corner solution.
  Try to formulate a “one additional dollar” argument yourself.

Utility Maximizations without Derivatives

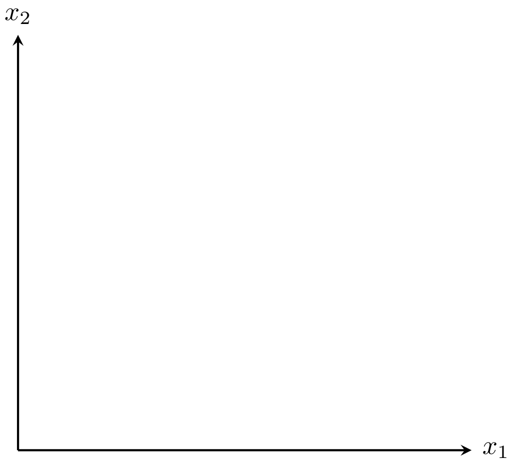
Perfect Complements

Consider the following utility function:

$$u(x_1, x_2) = \min\{x_1, x_2\}.$$

That is, increasing x_1 only, or x_2 only does not increase the utilities. Instead, the two goods must be consumed in pairs to increase utilities. We call x_1 and x_2 are *perfect complements*.

Can you draw the indifference curves?



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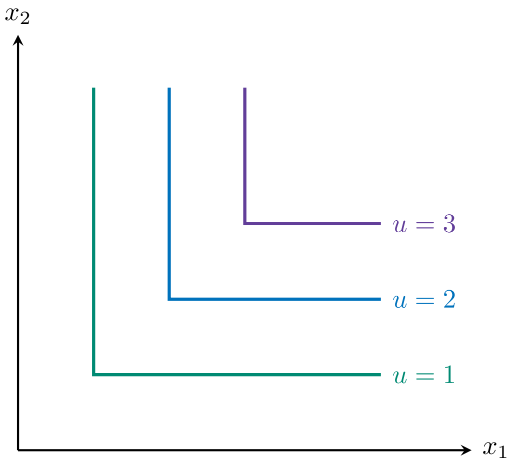

Perfect Complements

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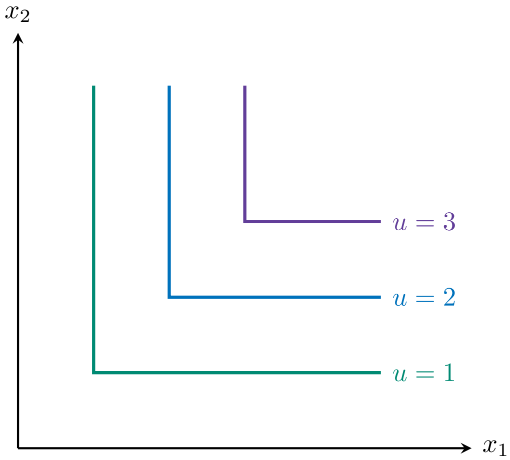

“MRS” under Perfect Complements

$u(x_1, x_2) = \min\{x_1, x_2\}$ is obviously not differentiable. Try the definition: in order to keep the utility level unchanged, how many units of x_2 is needed to substitute 1 unit of x_1?

- If $x_1 > x_2$:

- If $x_1 < x_2$:

“MRS” under Perfect Complements

$u(x_1, x_2) = \min\{x_1, x_2\}$ is obviously not differentiable. Try the definition: in order to keep the utility level unchanged, how many units of x_2 is needed to substitute 1 unit of x_1?

- If $x_1 > x_2$: Slightly reducing x_1 does not really change the utility level. No need to add x_2 to compensate reducing x_1.
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- If $x_1 > x_2$: Slightly reducing x_1 does not really change the utility level. No need to add x_2 to compensate reducing x_1. $|MRS(x_1, x_2)| = 0$.
- If $x_1 < x_2$:

“MRS” under Perfect Complements

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- If $x_1 < x_2$: Slightly reducing x_1 will reduce the utility level in the same scale, which cannot be compensated regardless how many units of x_2 are consumed.

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- If $x_1 < x_2$: Slightly reducing x_1 will reduce the utility level in the same scale, which cannot be compensated regardless how many units of x_2 are consumed. $|MRS(x_1, x_2)| = \infty$.

"MRS" under Perfect Complements

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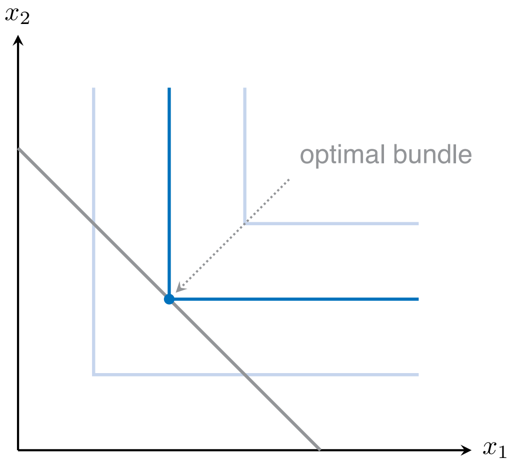
- If $x_1 < x_2$: Slightly reducing x_1 will reduce the utility level in the same scale, which cannot be compensated regardless how many units of x_2 are consumed. $|MRS(x_1, x_2)| = \infty$.

One cannot really substitute x_1 with x_2, vice versa. In other words, ideally, x_1 and x_2 should be consumed on pairs.

- Coffee and cream, left shoe and right shoe…
- Hence the name, perfect complements.

Utility Maximizations under Perfect Complements

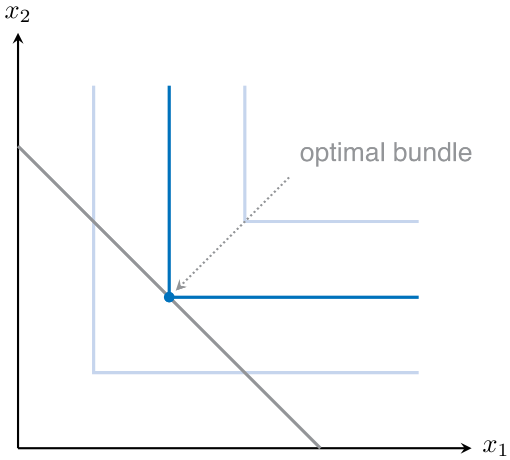
- The optimal bundle is always at the “kink” where x_1 and x_2 are purchased in pairs.



Utility Maximizations under Perfect Complements

- The optimal bundle is always at the “kink” where x_1 and x_2 are purchased in pairs.
- Demand functions:

$$x_1^\star(p_1, p_2, I) = x_2^\star(p_1, p_2, I) = \frac{I}{p_1 + p_2}.$$



Summary

Summary of Lecture 3

- For differentiable utilities, first consider Lagrange method:

$$\mathcal{L}(x_1, \cdots, x_n, I, \lambda) = u(x_1, \cdots, x_n) + \lambda(I - p_1 x_1 - \cdots - p_n x_n),$$

  which gives the utility maximization condition

$$|MRS(x_i, x_j)| = \frac{\partial u / \partial x_i}{\partial u / \partial x_j} = \frac{p_i}{p_j}.$$

- One may end up with corner solutions, where the condition above may not hold.
- Some utilities are not differentiable – the graphical method may help.
- Features of preferences: homotheticity, quasilinearity.
- Special types of preferences: perfect substitutes, perfect complements.
- Special utility functions: Cobb-Douglas, CES.